Implied Impermanent Loss for Concentrated Liquidity

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Yield Farming by Liquidity Provision

- Situation: Trade BTC for ETH on a Decentralized Exchange (DEX).

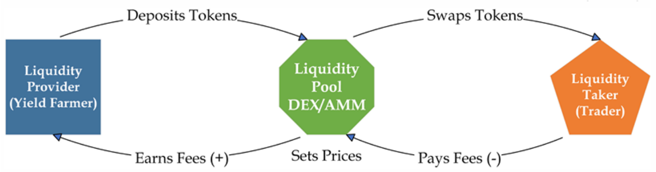


Figure: **Yield Farming via Liquidity Provision**

- In V2, AMM: $L = \sqrt{N_1 N_2}$, where L is the total liquidity, and N_1 and N_2 are the amounts of tokens 1 and 2.
- If a trader wants 1 unit of token 2 from the pool, the AMM requires Δ_1 units of token 1, where Δ_1 satisfies $\sqrt{(N_1 + \Delta_1)(N_2 - 1)} = L$.
- Impermanent Loss (IL): if the fundamental value of the tokens changes, an arbitrageur trades at the stale price in the direction of the price change, thus minimizing the pool's value (adverse selection). details

Example of Impermanent Loss (IL v2)

- Assume: Token A = 100 USD (USDT).
- The pool initially contains 900 Token A and 90,000 USDT (Relative Price = $90,000/900 = 100$).
- An LP with 100 Token A and 11,000 USDT adds 100 Token A and 10,000 USDT to the pool (keeping 1,000 USDT idle), obtaining a 10% share of a 200,000 USD pool
  - New pool size: $1,000$ Token A $\times 100 + 100,000 = 200,000$ USD.
- Assume Token A increases from 100 USD to 121 USD. Traders will add USDT and remove Token A until the new ratio is 121:1.
- Hence $x/y = 121$, and $x \times y = 100,000,000$.
- 110,000 USDT and 909.1 Token A (110,000 × 909.1 = 100,000,000 and 110,000 / 909.1 = 121).
- Withdrawing the 10% yields 11,000 USDT and 90.9 Token A.
- The value of the share is $\approx 22,000$ USD + starting $1,000 = 23,000$.
- HODL: $11,000$ USD + 100×121 USD $= 23,100$ USD.
- IL $\approx$ 100 USD (vs. fee revenue from trading).

Example of Impermanent Loss (IL v3) - Concentrated Liquidity

- Assume: Token A, $S_0 = 100$ USD (USDT as the stablecoin),
- Range for the LP to supply liquidity: $[S_l, S_u] = [81, 121]$ USD,
- At time t_0: the LP owns $x_0 = 100$ Token A worth 10,000 USD at the price $S_0 = 100$,
- Following the Uniswap maths (details) for token balances:
  - $x_0 = 100$ Token A and the USDT position $y_0 = 11,000$.
- Time t_0: $V_0 = x_0 S_0 + y_0 = 100 \times 100 + 11,000 = 21,000$ USD.
- Time t_1: the market price of Token A increases from 100 USD to 121 USD.

Holding

- 100 Token A $\times\, 121 = 12,100$ USD
- $+$ initially held $11,000$ USDT
- $= V_{\text{hold}}(S_1) = 23,100$ USD.

LP

- $S \to S_u$, position only in USDT.
- $y_{\text{pos}}(S_u) = 11,000 \times (11 - 9) = 22,000$ USDT.
- $V_{\text{LP}}(S_1) = 0 \times 121 + 22,000$ USDT.

The IL is therefore $V_{\text{hold}}(S_1) - V_{\text{LP}}(S_1) = 23,100 - 22,000 = 1,100$ USD.

This Paper - Main Results

- Option-Implied Impermanent Loss (ILL) Analysis
  - Replication of IIL for 15 pools across maturities (7 to 30 days).
  - As expected, IIL for concentrated liquidity (i.e. "v3") is higher when liquidity is stronger concentrated.
  - Higher IIL in exotic pools (for example, XRP/SOL).
  - Increasing term structure for Vola-Vola v3 pools and flat for Vola-Stable pools.
  - Introduced Impermanent Loss Risk Premia (IIL - RIL) for Concentrated Liquidity.
  - ILRP is higher when liquidity is more concentrated.
- Mathematical Framework
  - Risk-neutral valuation of IIL v3, using a unified approach based on traded options.
  - Estimating the risk-neutral density as no options on the relative price are available.

Literature Review

This paper bridges the literature on the risks of decentralized liquidity provision and the option-implied pricing information.

1. Literature on Liquidity Provision on DEXs and IL
   Li et al. (2024), Heimbach et al. (2022), Milionis et al. (2024), Fukasawa et al. (2023), Cartea et al. (2024), Lehar and Parlour (2023), Clark (2020), Clark (2021)
2. Literature on Option-Implied Information (mostly for equities):
   IV/*VRP*: Carr and Wu (2009), Bollerslev et al. (2009), Drechsler and Yaron (2011), Bollerslev et al. (2014)
   IV_{dn}/IV_{up}: Feunou et al. (2018), Kilic and Shaliastovich (2019)
   IV/*VRP* for BTC: Alexander and Imeraj (2021)
3. *IIL v*2 for BTC-ETH (30 days)
   Papanicolaou et al. (2025)

Mathematical Framework

Impermanent Loss in Continuous Time (IIL v2)

- AMM with constant product rule and with (exogenous) prices (real-world dynamics) of tokens following

$$dP_i\left(t\right) = \mu_i P_i\left(t\right)dt + \sigma_i\left(t\right)P_i\left(t\right)dB_i\left(t\right), \quad i = 1,\ 2, \tag{1}$$

where $B_i(t)$ for $i = 1$ and 2 are two correlated Brownian motions.

- IL is the integrated variance of the relative price ($R(t) = P_1(t)/P_2(t)$) of the two tokens (details , Li et al. (2024), Milionis et al. (2024)),

$$dIL\left(t\right) = -\frac{1}{8}\sigma_R^2\left(t\right)dt, \tag{2}$$

where

$$\sigma_R\left(t\right) = \sqrt{\sigma_1^2\left(t\right) - 2\rho\sigma_1\left(t\right)\sigma_2\left(t\right) + \sigma_2^2\left(t\right)}\ . \tag{3}$$

- IL is a function of the tokens' variances and their correlation.

Implied Impermanent Loss (IIL v2) – Method & Challenges

- Goal: price IIL via a variance-swap representation on the relative price

$$R(t) = \frac{P_1(t)}{P_2(t)} \ .$$

- Key theoretical challenges:
  - $R(t)$ is not a $\mathbb{Q}$-martingale $\Rightarrow$ standard variance-swap pricing fails.
  - The log contract on $R(T)$ is not directly replicable since options trade only on individual tokens.
- Methodological solution (see Papanicolaou et al. (2025)):
  - Change of numéraire to a tilted measure $\widetilde{\mathbb{Q}}$ under which $R(T)$ is a martingale.
  - Construct a joint risk-neutral density for $(P_1(T), P_2(T))$ using options on individual tokens.
  - Use Hansen–Jagannathan bounds to identify the least-distorted joint pricing kernel.
- We obtain a model-free implied measure of impermanent loss,

$$\widetilde{\mathbb{E}}^{\mathbb{Q}}[IL(T)] = -\frac{1}{8T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\left[\int_0^T \sigma_R^2(t)\,dt\right] \ .$$

Implied Impermanent Loss (IIL v3) – I

- Under the Uniswap v3 protocol, LPs allocate tokens within a specific price range.
- Consider the ticks $(r_i)_{i=1,\,2,\,\dots,\,m}$ where $r_{i+1} = r_i \times 1.0001$, and let $C_i = [r_i,\ r_{i+1})$ denote the band.
- If $R(t) \in C_i$ then the LP earns a reward but is also exposed to IL.
- The IL dynamic analog is

$$dIL_i(t) = -\frac{\sigma_R^2(t)\sqrt{R(t)}}{4\left(2\sqrt{R(t)} - \sqrt{r_i} - \frac{R(t)}{\sqrt{r_{i+1}}}\right)} \mathbf{1}_{\{R(t)\in C_i\}} dt\ . \tag{4}$$

- This increment of impermanent loss can be derived using Itô's lemma as done in deriving the dynamic for IIL v2.

Implied Impermanent Loss (IIL v3) – II

- Integrating over time yields the total impermanent loss, which is equivalent to a corridor variance swap (Lee (2010a) and Lee (2010b)),

$$\begin{aligned}\widetilde{\mathbb{E}}^{Q}\left[IL_i\left(T\right)\right] &= -\frac{1}{4}\widetilde{\mathbb{E}}^{Q}\left[\int_0^T \frac{\sigma_R^2\left(t\right)\sqrt{R\left(t\right)}}{2\sqrt{R\left(t\right)}-\sqrt{r_i}-\frac{R(t)}{\sqrt{r_{i+1}}}}\mathbf{1}_{\{R(t)\in C_i\}}\left(t\right)dt\right], \qquad (5)\\ &= -\frac{1}{2}\int_{K\in C_i}\frac{P(K)\mathbf{1}_{\{K<R(0)\}}+C(K)\mathbf{1}_{\{K\geq R(0)\}}}{\sqrt{K^3}\left(2\sqrt{K}-\sqrt{r_i}-\frac{K}{\sqrt{r_{i+1}}}\right)}dK\end{aligned}$$

  where $P(K)=\widetilde{\mathbb{E}}^{Q}\left[(K-R\left(T\right))^{+}\right]$ and $C(K)=\widetilde{\mathbb{E}}^{Q}\left[(R(T)-K)^{+}\right]$.
- We can evaluate IIL v3 with the same Risk Neutral Density we constructed for IIL v2 (Li et al. (2024)).

Implied Impermanent Loss (IIL v3) – III

- We define liquidity ranges using a concentration parameter α.
- Let $S_t = P_{1,t}/P_{2,t}$ denote the relative price at time t.
- We consider three different "ranges":
  - Symmetric

$$S_{\ell,t} = (1-\alpha)S_{t-1}, \qquad S_{u,t} = (1+\alpha)S_{t-1} \ . \tag{6}$$

  - Up (only provide liquidity for increasing prices)

$$S_{\ell,t} = S_{t-1}(1-0.01) \qquad S_{u,t} = (1+10)S_{t-1} \ . \tag{7}$$

  - Dn (only provide liquidity for decreasing prices)

$$S_{\ell,t} = S_{t-1}10^{-4} \qquad S_{u,t} = S_{t-1}(1+0.01) \ . \tag{8}$$

- We use these ranges to evaluate both IIL and RIL.

Implied Impermanent Loss (IIL v3) – IV

Given the liquidity ranges $[S_{\ell,t}, S_{u,t}]$, we estimate both implied and realized impermanent loss.

- IIL
  - For v3, we compute the IIL by integrating over S_ℓ and S_u the IL price

$$IIL^{v3} = \frac{1}{T/365} \int_{S_\ell}^{S_u} IL(K)\, dK \ . \tag{9}$$

  - For v2, we follow Papanicolaou et al. (2025) and compute

$$IIL^{v2} = \widetilde{\mathbb{E}}^{Q}\left[\text{IL}\,(T)\right] = -\frac{\mathbb{E}^{Q}\left[P_2(T)\ln\left(\frac{R(T)}{R(0)}\right)\right]}{4e^{rT}P_2(0)} .$$

- RIL
  - For v3, instantaneous IL (Heimbach et al. (2022)) is computed hourly using the same ranges:

$$RIL_t^{v3} = -IL\left(S_{t-1}, S_t, S_{\ell,t}, S_{u,t}, \tilde{L}\right) .$$

  - RIL is then aggregated over a rolling window of $T \cdot 24$ hours and annualized.
  - For v2, RIL is given by $1/8$ of the annualized variance of the price ratio.
- The impermanent loss risk premium is defined as

$$ILRP := IIL - RIL.$$

Empirical Analysis

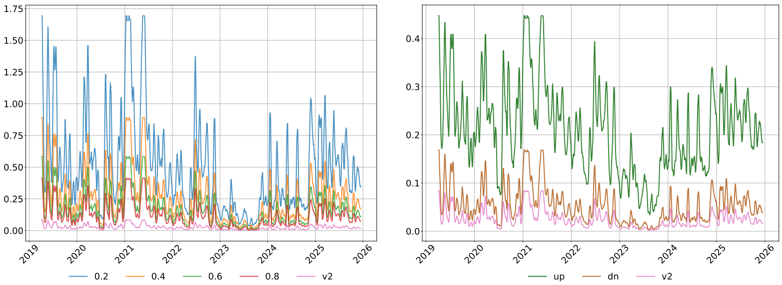
Options – Deribit

- Deribit: Largest (centralized) exchange for crypto options trading.
- Options on
  - BTC, ETH, XRP, SOL, BNB,
  - We focus on short maturities: 7, 14, 21, and 30 days,
  - Start dates: BTC & ETH (April 2019), XRP & SOL (March 2024), BNB (October 2024).
- Surface data from Deribit (5×2 points) provided by Amberdata.io.
- We calculate the univariate RNDs following Breeden and Litzenberger (1978)

$$\frac{\partial^2 C}{\partial K^2}|_{K=x} = e^{-r\tau} f^Q(x). \tag{10}$$

- Pools:
  - Vola-Vola: both tokens are volatile (all ten tuples involving the five tokens)
  - Vola-Stable: volatile token and a stablecoin, such as BTC-USD, ETH-USD, ...
- We estimate the IIL, RIL, and ILRP for v2 and v3 for all pools and maturities.

Implied Impermanent Loss (IIL v3) – I



(a) *IIL* for $\alpha = 0.2, 0.4, 0.6, 0.8$ and v2 (b) IIL for α = up, down, and v2

Figure: **IIL for BTC-ETH across Ranges.** The figure reports the implied impermanent loss IIL^{v3} for the BTC-ETH pool at a 7-day maturity. The left column shows $\alpha = 0.2, 0.4, 0.6, 0.8$ and the v2 benchmark, while the right column compares the up, down, and v2 ranges.

- IIL increases when liquidity is concentrated: a smaller range increases v3 leverage.
- Large IIL up: When prices go up, LPs gradually sell the rising token, and opportunity costs become larger.

Implied Impermanent Loss (IIL v3) – II

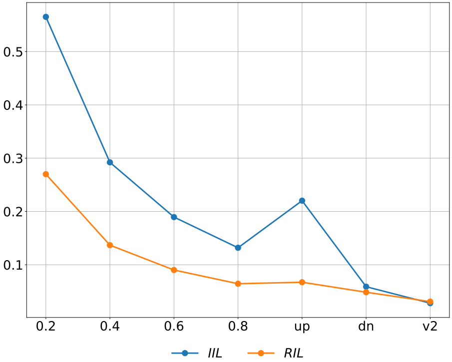
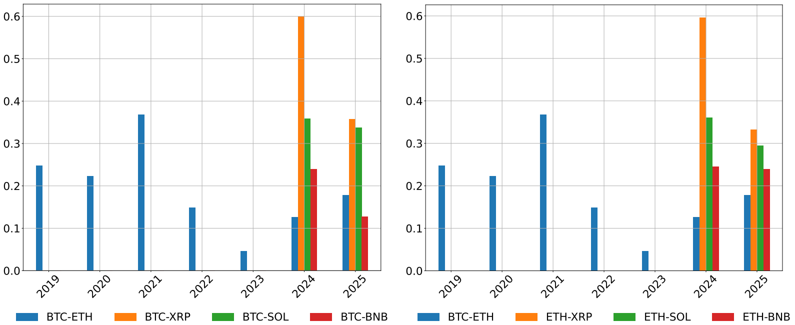


Figure: **ILL for BTC-ETH pool across Ranges.** The figure reports the average for the IIL for the BTC-ETH pool across Ranges at a 7-day maturity..

- IIL is higher when liquidity is concentrated.
- IIL is on average above RIL, except for the v2 range (similar).

Implied Impermanent Loss (IIL v3) – III



(a) $IIL^{0.6}$: BTC-Vola

(b) $IIL^{0.6}$: ETH-Vola

Figure: **IIL for BTC and ETH Pools, 60% Range.** The figure reports the average for the implied impermanent loss $IIL^{0.6}$ for the BTC and ETH pools at a 7-day maturity.

- Exotic pools (e.g., ETH–XRP, BTC-SOL, ETH–SOL) show higher IIL compared to pairs like BTC–ETH or BTC-BNB.
- IIL reached its lowest point in 2023, but has risen noticeably in 2024–2025.

Implied Impermanent Loss (IIL v3) – IV

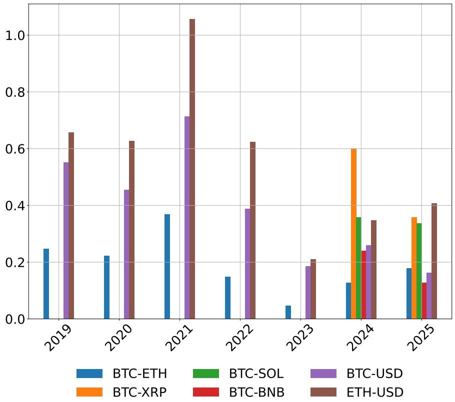
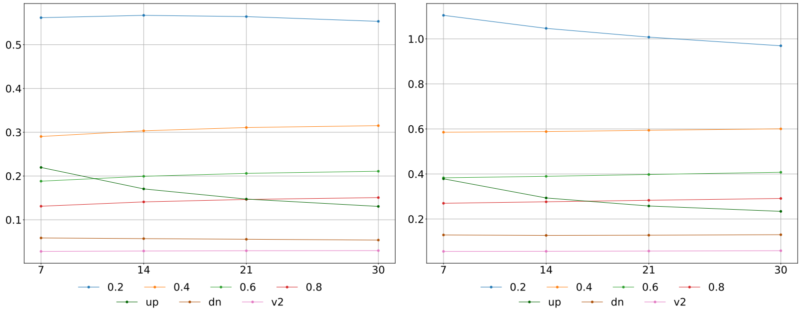


Figure: **IIL for BTC Pools: Vola–Vola (60% Range).** The figure reports the average for the implied impermanent loss $IIL^{0.6}$ for the BTC pools at a 7-day maturity.

- Vola–Stable pools usually show higher IIL.
- IIL is higher for Vola-Stable pools because in Vola-Vola pools, individual tokens move together and cause less impermanent loss

Implied Impermanent Loss (IIL v3) – V



(a) Term structure - BTC-ETH

(b) Term structure - BTC-USD

Figure: **Term structure of** *IIL*. This figure shows the term structure of the *IIL* for the BTC-ETH pool (left panel) and for BTC-USD (right panel).

- BTC-ETH: upward sloping term structure.
- BTC-USD: flat term structure.
- When liquidity is tight (or tilted upward), most of the IL risk is about near-term price moves. That's why the IIL is larger at 7 days for $\alpha = 0.2$, "up".

Implied Impermanent Loss (IIL v3) – VI

Series	α	BTC–ETH Mean	BTC–ETH $Q^{0.1\%}$	BTC–USD Mean	BTC–USD $Q^{0.1\%}$
ILRP	0.2	0.295	-0.011	0.555	-6.916
ILRP	0.4	0.155	-0.000	0.307	-2.689
ILRP	0.6	0.099	-0.004	0.199	-1.715
ILRP	0.8	0.067	-0.006	0.139	-1.224
ILRP	up	0.153	0.022	0.252	-0.623
ILRP	dn	0.010	-0.046	0.037	-0.846
ILRP	v2	-0.003	-0.083	-0.007	-1.133

Table: **Average and $Q^{0.1\%}$ ILRP values for BTC-ETH and BTC-USD (7 days) across Ranges.**

- Higher average ILRP when liquidity is more concentrated.
- During downturns, tokens in Vola-Vola pools often decline together, implying lower IL. In contrast, Vola–Stable pools exhibit stronger price divergence, leading to higher IL.
- ILRP gets negative in market turmoils (similar to VRP for equities).

Conclusion

- Mathematical Framework and Estimation:
  - Extend Risk-Neutral ("Implied") Impermanent Loss (IIL) for v3.
  - Density via HJ bounds (no index volatility available).
- Empirically
  - Computed IIL for v3 from traded derivatives for 15 pools, various maturities, and for different ranges.
  - IIL is higher for concentrated ranges of liquidity.
  - Higher risk in exotic pools.
  - Increasing term structure for Vola-Vola pools and flat for Vola-Stable pools.
  - Impermanent Loss Risk Premia (IIL - RIL) is larger when liquidity is concentrated.

$\Rightarrow$ **Many open questions to be answered!**

$t^h a_n (k)\ \ y_o [u]$!

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Appendix

Mathematical Framework

Continuous-Time Approach – Single Pool – AMM

- The constant product rule for a LP ($N_1(t)$ and $N_2(t)$ amount of tokens):

$$L := \sqrt{N_1(t)N_2(t)}. \tag{11}$$

- From this constant product rule (details) (11), a relative price emerges:

$$R(t) = \frac{N_2(t)}{N_1(t)} = \text{token 2 per token 1}, \tag{12}$$

- In terms of absolute price, we assume

$$R(t) = \frac{P_1(t)}{P_2(t)} = \frac{\text{dollar per token 1}}{\text{dollar per token 2}} = \text{token 2 per token 1}, \tag{13}$$

where $P_i(t)$ for $i = 1$ and 2 are the exogenous prices of tokens.

Single Pool – Token Dynamics

- The prices of tokens in the LP follow GBMs (real-world dynamics)

$$dP_i(t) = \mu_i P_i(t)dt + \sigma_i P_i(t)dB_i(t), \quad i = 1,\ 2, \tag{14}$$

  where $B_i(t)$ for $i = 1$ and 2 are two correlated Brownian motions.
- Apply Itô to get the SDE for the relative price $R(t) = P_1(t)/P_2(t)$ and:

$$dR(t) = d\left(\frac{P_1(t)}{P_2(t)}\right) = \cdots = \mu_R(t)R(t)dt + \sigma_R(t)R(t)dB_R(t),$$

$$dN_1(t) = d\left(\frac{L}{\sqrt{R(T)}}\right), \tag{15}$$

$$dN_2(t) = d(L\sqrt{R(T)}) \tag{16}$$

where $\sigma_R^2 = \sigma_1^2(t) - 2\rho\sigma_1(t)\sigma_2(t) + \sigma_2^2(t)$, $\mu_R(t) = \mu_1 - \mu_2 + \sigma_2^2(t) - \rho\sigma_1(t)\sigma_2(t)$.

Single Pool – Impermanent Loss

- ... which allows us to derive the expression for the impermanent loss

$$dIL(t) := \frac{\mathcal{V}_{t+dt}^{\text{staked}} - \mathcal{V}_{t+dt}^{\text{held}}}{\mathcal{V}_{t+dt}^{\text{held}}} = \cdots = -\frac{\sigma_R^2(t)}{8} dt \leq 0, \quad \forall t \geq 0, \tag{17}$$

- impermanent loss $\approx$ realized volatility of the relative price (R)
- $\sigma_R^2 = \sigma_1^2(t) - 2\rho\sigma_1(t)\sigma_2(t) + \sigma_2^2(t)$
- $dIL(t)$ more negative for high variances and high negative correlation
- $dIL(t) \approx 0$ if $\sigma_1^2 = \sigma_2^2(t) \approx 0$, or $\sigma_1^2(t) = \sigma_2^2(t)$, $\rho = 1$
- details
- See Li et al. (2024) for a rigorous derivation.

Impermanent Loss

Impermanent Loss (IL)

- IL: Opportunity-cost dynamics between providing liquidity and holding the underlying tokens to potentially profit from the price movement.
- IL: prices of tokens diverge (no matter in which direction), causing LPs to underperform a basic buy-and-hold strategy.
  - IL $\neq$ LPs experiences a negative return:
    asset value from buy-and-hold $\geq$ asset value from the liquidity provision.

Impermanent Loss (IL)

$$dI_t := \frac{d\left(N_1(t)P_1(t) + N_2(t)P_2(t)\right) - \left(N_1(t)dP_1(t) + N_2(t)dP_2(t)\right)}{N_1(t)P_1(t) + N_2(t)P_2(t)} \tag{18}$$

$$= -\frac{\sigma_R^2}{8}dt \leq 0, \quad \forall t \geq 0, \tag{19}$$

Implied Impermanent Loss

Change of Measure – I

Recall our pair of SDEs for token prices,

$$dP_1(t) = rP_1(t)dt + \sigma_1(t)P_1(t)dB_1(t)$$
$$dP_2(t) = rP_2(t)dt + \sigma_2(t)P_2(t)dB_2(t) \ ,$$

where $B_1(t)$ and $B_2(t)$ are risk-neutral standard Brownian motions with correlation parameter ρ. Here, we let $\mathbb{Q}$ denote the risk-neutral measure.

Change of Measure – II

$$\begin{aligned}
\text{Call}^{spr}(T,K) &= e^{-rT}\mathbb{E}^{\mathbb{Q}}(P_1(T)-KP_2(T))^+ && (20)\\
&= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left(P_2(T)(R(T)-K)^+\right) && (21)\\
&= P_2(0)\mathbb{E}^{\mathbb{Q}}\left(\frac{P_2(T)}{P_2(0)e^{rT}}(R(T)-K)^+\right) && (22)\\
&= P_2(0)\widetilde{\mathbb{E}}^{\mathbb{Q}}(R(T)-K)^+ \ , && (23)
\end{aligned}$$

where $\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|_T = \frac{P_2(T)}{P_2(0)e^{rT}}$, under which $R(T)$ is a $\widetilde{\mathbb{Q}}$ martingale.

$$\begin{aligned}
\widetilde{\mathbb{E}}^{\mathbb{Q}}[\frac{P_1(T)}{P_2(T)}] &= \mathbb{E}^{\mathbb{Q}}[\frac{P_2(T)}{P_2(0)e^{rT}}\frac{P_1(T)}{P_2(T)}] = \mathbb{E}^{\mathbb{Q}}[\frac{P_1(T)}{P_2(0)e^{rT}}] = \frac{\mathbb{E}^{\mathbb{Q}}[P_1(T)]}{P_2(0)e^{rT}} && (24)\\
&= \frac{P_1(0)e^{rT}}{P_2(0)e^{rT}} = \frac{P_1(0)}{P_2(0)} && (25)
\end{aligned}$$

Change of Measure – Girsanov – I

Let

$$X(t) = \int_0^t \sigma_2(s) dB_2^{\mathbb{Q}}(s),$$

which is a martingale under $\mathbb{Q}$. The Dolean-Dade exponent is

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}}\Big|_T = \mathcal{E}(X(T)) = e^{X(T) - \frac{1}{2}\int_0^T \sigma_2^2(t)dt} = \frac{P_2(T)}{e^{rT} P_2(0)} \ .$$

Girsanov Theorem: Given $W(t)$ that is Brownian motion under $\mathbb{Q}$, we define

$$\widetilde{W}^{\mathbb{Q}}(t) := W^{\mathbb{Q}}(t) - [W^{\mathbb{Q}}, X](t) \ ,$$

which is Brownian motion under $\widetilde{\mathbb{Q}}$.

- Hence, $\widetilde{B}_2^{\mathbb{Q}}(t) = B_2^{\mathbb{Q}}(t) - \int_0^t \sigma_2(s) ds$ is $\widetilde{\mathbb{Q}}$ Brownian motion.

Change of Measure – Girsanov – II

Suppose for $\mathbb{Q}$ Brownian motion $W(t)$ that $dB_2^{\mathbb{Q}}(t)dW(t) = \rho dt$. Then by Girsanov theorem, define

$$\widetilde{W}^{\mathbb{Q}}(t) = W^{\mathbb{Q}}(t) - [W, X](t) = W^{\mathbb{Q}}(t) - \rho \int_0^t \sigma_2(s)ds \ ,$$

which is $\widetilde{\mathbb{Q}}$ Brownian motion. In particular, for the token prices P_1 and P_2, under the new measure we have

$$\begin{aligned} dP_1(T) &= rP_1(T)dt + \sigma_1(t)P_1(T)\left(dB_1^{\mathbb{Q}}(t) - \rho\sigma_2(t)dt + \rho\sigma_2(t)dt\right) \\ &= \left(r + \rho\sigma_1(t)\sigma_2(t)\right)P_1(T)dt + \sigma_1(t)P_1(T)d\widetilde{B}_1^{\mathbb{Q}}(t) \\ dP_2(T) &= \left(r + \sigma_2^2(t)\right)P_2(T)dt + \sigma_2(t)P_2(T)d\widetilde{B}_2^{\mathbb{Q}}(t) \end{aligned}$$

where $\widetilde{B}_1^{\mathbb{Q}}(t) = B_1^{\mathbb{Q}}(t) - \rho\int_0^t \sigma_2(s)ds$. Notice that $e^{-rt}/P_2(T)$ is a $\widetilde{\mathbb{Q}}$ martingale (i.e., the discounted price of the original 'currency' is a martingale under the change of numeraire).

Implied Impermanent Loss – Details

Remember: Volatility Swap (Demeterfi et al. (1999); Carr and Madan (1998); Bakshi et al. (2015)):

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{1}{T}\int_0^T \sigma_i^2(t)dt\right] = \frac{2}{T}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{dP_i(t)}{P_i(t)}\right] - \frac{2}{T}\mathbb{E}^{\mathbb{Q}}\log\left(\frac{P_i(T)}{P_i(0)}\right) \quad \text{for } i = 1, 2. \tag{26}$$

Then for the log-contracts (for P_1 and P_2 under tilde measure and $r = 0$),

$$\frac{2}{T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\log(P_1(T)/P_1(0)) = \frac{2}{T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\int_0^T\left(\rho\sigma_1(t)\sigma_2(t) - \frac{1}{2}\sigma_1(t)^2\right)dt$$

$$\frac{2}{T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\log(P_2(T)/P_2(0)) = \frac{1}{T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\int_0^T \sigma_2(t)^2 dt \ .$$

Implied Impermanent Loss – III

Which we put together to obtain the $\widetilde{\mathbb{Q}}$-valuation of the IL,

$$\widetilde{\mathbb{E}}^{\mathbb{Q}}\mathrm{IL}_T = -\frac{1}{8T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\int_0^T \left(\sigma_1(t)^2 + \sigma_2(t)^2 - 2\rho\sigma_1(t)\sigma_2(t)\right) dt \tag{27}$$

$$= -\frac{1}{4T}\left(\widetilde{\mathbb{E}}^{\mathbb{Q}}\log(P_2(T)/P_2(0)) - \widetilde{\mathbb{E}}^{\mathbb{Q}}\log(P_1(T)/P_1(0))\right) \tag{28}$$

$$= \frac{1}{4T}\widetilde{\mathbb{E}}^{\mathbb{Q}}\log(R(T)/R(0)), \tag{29}$$

where we took advantage of the logarithm property

$$\log\left(\frac{R_T}{R_0}\right) = \log\left(\frac{P_1(T)}{P_1(0)}\right) - \log\left(\frac{P_2(T)}{P_2(0)}\right). \tag{30}$$

Implied Impermanent Loss – v3

- We have that $dR(t) = \mu_R R(t)dt + \sigma_R(t)R(t)dW_R(t)$. And,

$$
\begin{aligned}
& R_{t+\Delta t}\mathbf{1}_{R_{t+\Delta t}<R_l} - R_{t+\Delta t}\mathbf{1}_{R_t<R_l} \\
& \approx (R_t + dR_t)(\mathbf{1}_{R_t<R_l} - \delta_{R_l}(R_t)dR_t - \frac{1}{2}\delta'_{R_l}(R_t)(dR_t)^2) - (R_t + dR_t)\mathbf{1}_{R_t<R_l} \\
& = -\delta_{R_l}(R_t)(R_t dR_t + (dR_t)^2) - \frac{1}{2}\delta'_{R_l}(R_t)R_t(dR_t)^2
\end{aligned}
$$

- Then,

$$
\begin{aligned}
& \mathbf{1}_{R_{t+\Delta t}>R_u} - \mathbf{1}_{R_t>R_u} \\
& \approx \mathbf{1}_{R_t>R_u} + \delta_{R_u}(R_t)dR_t + \frac{1}{2}\delta'_{R_u}(R_t)(dR_t)^2 - \mathbf{1}_{R_t>R_u} \\
& = \delta_{R_u}(R_t)dR_t + \frac{1}{2}\delta'_{R_u}(R_t)(dR_t)^2
\end{aligned}
$$

Implied Impermanent Loss – v3

- Finally

$$
\begin{aligned}
&\left(2\sqrt{R_{t+\Delta t}} - \sqrt{R_l} - \frac{R_{t+\Delta t}}{\sqrt{R_u}}\right)\mathbf{1}_{R_l \le R_{t+\Delta t} < R_u} - \left(\frac{R_t + R_{t+\Delta t}}{\sqrt{R_t}} - \sqrt{R_l} - \frac{R_{t+\Delta t}}{\sqrt{R_u}}\right)\mathbf{1}_{R_l \le R_t < R_u} \\
&\approx \left(2\sqrt{R_t} + \frac{dR_t}{\sqrt{R_t}} - \frac{(dR_t)^2}{4R_t^{3/2}} - \sqrt{R_l} - \frac{R_t + dR_t}{\sqrt{R_u}}\right) \\
&\quad \times \left(\mathbf{1}_{R_l \le R_t < R_u} + (\delta_{R_l}(R_t) - \delta_{R_u}(R_t))dR_t + \frac{1}{2}(\delta'_{R_l}(R_t) - \delta'_{R_u}(R_t))(dR_t)^2\right) \\
&- \left(\frac{2R_t + dR_t}{\sqrt{R_t}} - \sqrt{R_l} - \frac{R_t + dR_t}{\sqrt{R_u}}\right)\mathbf{1}_{R_l \le R_t < R_u}
\end{aligned}
$$

Implied Impermanent Loss – v3

- The LVR in the band is, after a little bit of algebra

$$-\frac{\sigma_R^2(t)}{4}\sqrt{R_t}\mathbf{1}_{R_l \le R_t < R_u} dt \ .$$

- The IL is

$$dIL_t = -\frac{\sigma_R^2(t)}{4\left(2\sqrt{R_t} - \sqrt{R_l} - \frac{R_t}{\sqrt{R_u}}\right)}\sqrt{R_t}\mathbf{1}_{R_l \le R_t < R_u} dt \ .$$

Example of Impermanent Loss (IIL v3) - price range

Let's consider this example, where

- Assume: Token A, $S_0 = 100$ USD (USDT as the stablecoin),
- Range for the LP to supply liquidity: $[S_l, S_u] = [81, 121]$ USD,
- Let's define $\quad p = \sqrt{S} = 10, \quad a = \sqrt{S_l} = 9, \quad b = \sqrt{S_u} = 11,$
- At time t_0: the LP owns $x_0 = 100$ Token A worth 10,000 USD at the price $S_0 = 100$,
- With `liquidity` L and `price` $S \in [S_l, S_u]$ the token balances are

$$\text{Token A} = x(S) = L\frac{b-p}{pb} \quad \text{USDT} = y(S) = L(p-a)\;. \tag{31}$$

- Imposing that the position contains exactly $x_0 = 100$ Token A at S_0 implies $L = 11,000$. Hence, the corresponding USDT position at S_0 is

$$y_0 = L(p-a) = 11,000 \times (10-9) = 11,000 \text{ USDT}. \tag{32}$$

- Time t_0: initial portfolio value $V_0 = x_0 S_0 + y_0 = 100 \times 100 + 11,000 = 21,000$ USD.
- Time t_1: the market price of Token A increases from 100 USD to 121 USD.

Holding

- 100 Token A $\times\, 121 = 12,100$ USD
- $+$ initially held $11,000$ USDT
- $= V_{\text{hold}}(S_1) = 23,100$ USD.

LP

- $S \to S_u$, we have a position only in USDT.
- $y_{\text{pos}}(S_u) = 11,000 \times (11-9) = 22,000$ USDT.
- $V_{\text{LP}}(S_1) = 0 \times 121 + 22,000$ USDT.

The impermanent loss is therefore $V_{\text{hold}}(S_1) - V_{\text{LP}}(S_1) = 23,100 - 22,000 = 1,100$ USD.

Implied Impermanent Loss (IIL v3) – III

- Let $P_{1,t}$ and $P_{2,t}$ denote the prices of tokens 1 and 2 at time t, and define the price ratio $S_t = P_{1,t}/P_{2,t}$.
- For a liquidity range $[S_\ell, S_u]$ and normalized liquidity $\tilde{L} = 1$, we discretize the admissible price support using n grid points, $K \in [S_\ell, S_u]$,
- and compute IL payoffs for deviations of the future price ratio R from each grid point

$$\Pi(K, S) = \mathbf{1}_{\{K<S_0\}} \max(K - S, 0) + \mathbf{1}_{\{K \geq S_0\}} \max(S - K, 0) \ . \tag{33}$$

- Let $\tilde{q}(R)$ denote the estimated pricing kernel (or state-price density) of the future price ratio.
- For each grid point K, the impermanent loss price is computed as

$$IL(K) = \frac{\int \Pi(K, S)\, \tilde{q}(S)\, dS}{K^{3/2}\left(2\sqrt{K} - \sqrt{S_\ell} - \frac{K}{\sqrt{S_u}}\right)} \ , \tag{34}$$

- and we define the implied impermanent loss for v3 as

$$IIL^{v3} = \frac{1}{T/365} \int_{S_\ell}^{S_u} IL(K)\, dK \ . \tag{35}$$

Implied Impermanent Loss (IIL v2 and v3)

We consider a range $[S_{\ell,t}, S_{u,t}]$ using α as concentration parameter

- For $\alpha \in (0,1)$, we consider symmetric ranges

$$S_{\ell,t} = (1-\alpha)S_{t-1}\ , \qquad S_{u,t} = (1+\alpha)S_{t-1}\ . \tag{36}$$

- An upward-only range (up) with $S_{\ell,t} = S_{t-1}(1-0.01)$ and $S_{u,t} = (1+10)S_{t-1}$.
- A downward-only range (dn) with $S_{\ell,t} = S_{t-1}10^{-4}$ and $S_{u,t} = S_{t-1}(1+0.01)$.
- For the IIL of v2, we follow Papanicolaou et al. (2025), calculating the following equation

$$\mathrm{IIL}^{v2} = \widetilde{\mathbb{E}}^{Q}\left[\mathrm{IL}\left(T\right)\right] = -\frac{\mathbb{E}^{Q}\left[P_2\left(T\right)\ln\left(\frac{R(T)}{R(0)}\right)\right]}{4e^{rT}P_2\left(0\right)}\ . \tag{37}$$

- The resulting time series $IIL_t^{v3,\alpha}(T)$ is computed for each pool, maturity, and liquidity concentration parameter α.

Realized Impermanent Loss (RIL v2 and v3)

- The instantaneous IL is computed as

$$IL_t = IL\Big(S_{t-1}, S_t, S_{\ell,t}, S_{u,t}\ , \tilde{L}\Big)\ , \tag{38}$$

  where $IL(\cdot)$ denotes the Uniswap v3 IL function as outlined in Heimbach et al. (2022).
- We define realized impermanent loss at time t as

$$RIL_t^{v3} = -IL_t\ .$$

- For each maturity T, the RIL is aggregated over a rolling window of $T \cdot 24$ hourly observations using a rolling mean,

$$RIL_t^{v3}(T) = (24 \cdot 365) \cdot \frac{1}{(T \cdot 24)} \sum_{k=0}^{T \cdot 24 - 1} RIL_{t-k}^{v3}\ , \tag{39}$$

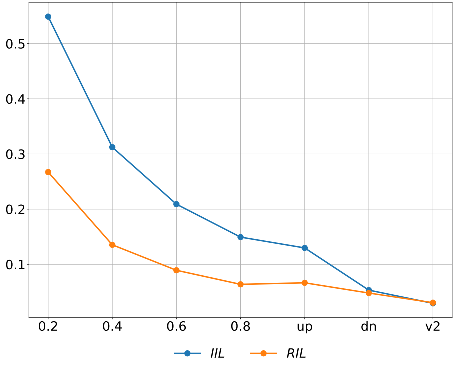
  where the window is evaluated using at least $T \cdot 24/2$ available observations.
- The RIL for v2 is simply 1/8 times the annualized variance of the relative price and is calculated following

$$RIL_t(T) = \frac{1}{8} \cdot (24 \cdot 365) \cdot \frac{1}{(T \cdot 24)}, \left(\sum_{k=0}^{T \cdot 24 - 1} \left(\Delta \log \left(\frac{P_{1,t-k}}{P_{2,t-k}} \right)^{(i,j)} \right)^2 \right)\ .$$

- The resulting time series $RIL_t^{v3,\alpha}(T)$ is computed for each pool, maturity, and liquidity concentration parameter α.
- The impermanent loss risk premia (ILRP) is defined as ILRP := IIL − RIL.

Implied Impermanent Loss (IIL v3) – I

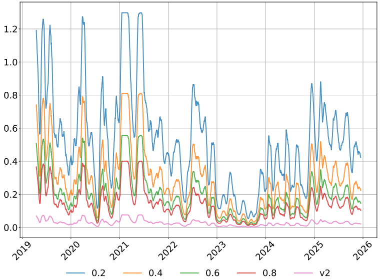
[label=Examplev3]



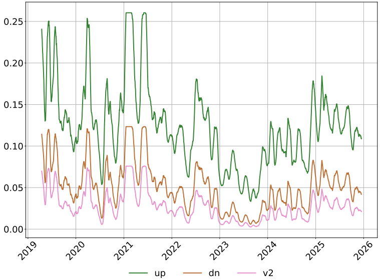
(a) Average BTC-ETH

Figure: **Average ILL for the BTC-ETH pool across Ranges - 30 days**.

Implied Impermanent Loss (IIL v3) – II



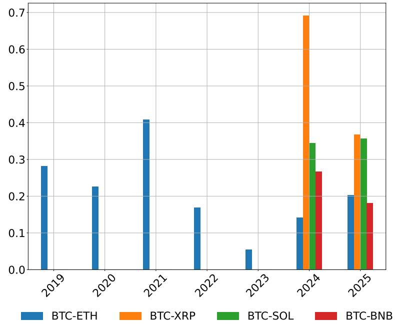
(a) *IIL* for $\alpha = 0.2, 0.4, 0.6, 0.8$ and v2



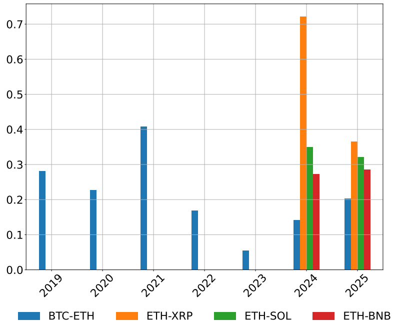
(b) IIL for $\alpha =$ up, down, and v2

Figure: **IIL for BTC-ETH across Ranges.** The figure reports the implied impermanent loss IIL^{v3} for the BTC-ETH pool at a 30-day maturity. The left column shows $\alpha = 0.2, 0.4, 0.6, 0.8$ and the v2 benchmark, while the right column compares the up, down, and v2 ranges. The data are sampled daily over April 2019–December 2025 and smoothed using a 14-day rolling mean.

Implied Impermanent Loss (IIL v3) – III



(a) $IIL^{0.6}$: BTC-Vola



(b) $IIL^{0.6}$: ETH-Vola

Figure: **IIL for BTC and ETH Pools (60% Range).** The figure reports the average for the implied impermanent loss $IIL^{0.6}$ for the BTC and ETH pools at a 30-day maturity.

Implied Impermanent Loss (IIL v3) – IV

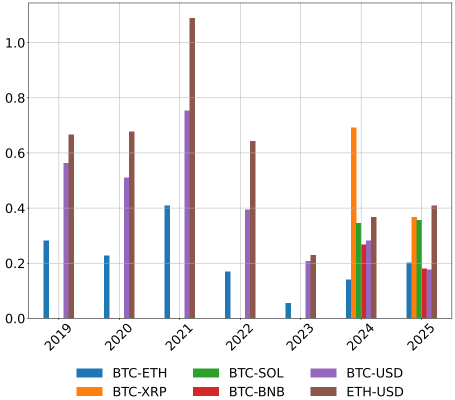
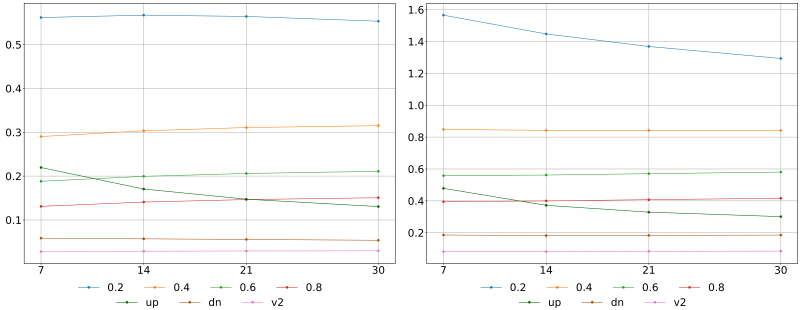


Figure: **IIL for BTC Pools: Vola–Vola (60% Range).** The figure reports the average for the implied impermanent loss $IIL^{0.6}$ for the BTC pools at a 30-day maturity.

Implied Impermanent Loss (IIL v3) – V



(a) Term structure - BTC-ETH

(b) Term structure - ETH-USD

Figure: **Term structure of** ***IIL***. This figure shows the maturity profile of the *IIL* for a Vola-Vola BTC-ETH pool (left panel) and for a Vola-Stable ETH pool (right panel).

Implied Impermanent Loss (IIL v3) – VI

		BTC–ETH		BTC–USD	
Series	α	Mean	$Q^{0.1\%}$	Mean	$Q^{0.1\%}$
ILRP	0.2	0.282	-0.055	0.419	-2.863
ILRP	0.4	0.177	-0.020	0.322	-1.131
ILRP	0.6	0.120	-0.013	0.224	-0.690
ILRP	0.8	0.086	-0.010	0.160	-0.488
ILRP	up	0.063	-0.009	0.107	-0.366
ILRP	dn	0.005	-0.057	0.038	-0.347
ILRP	v2	-0.001	-0.043	-0.004	-0.400

Table: **Average and $Q^{0.1\%}$ ILRP values for BTC-ETH and BTC-USD (30 days) across Ranges.**

Optimization Details

Joint Distribution – HJ Bounds

Hansen and Jagannathan (1991) (HJ): For any excess return Π and a pricing kernel (M) we have

$$0 = \mathbb{E}^{\mathbb{P}}[\Pi M] = \text{cov}(\Pi M) + \mathbb{E}[\Pi] \geq -\sigma(\Pi)\sigma(M) + \mathbb{E}[\Pi]. \tag{40}$$

HJ bound on Sharpe Ratios:

$$\sup_{\Pi:\sigma(\Pi)>0} \frac{\mathbb{E}[\Pi]}{\sigma(\Pi)} \leq \inf_{M\in\mathcal{M}} \sigma(M) \tag{41}$$

- The right-hand side motivates us to find a pricing kernel that minimizes the HJ bound.
- $M_{ij} = \frac{q_{ij}}{p_{ij}}$, where q_{ij} (p_{ij}) is the risk-neutral (real-world) measure.
- $\mathbb{E}^{\mathbb{P}}[M] = \sum_{i,j} M_{ij} p_{ij} = 1$, and hence $\sigma^2(M) = \sum_{i,j}(M_{ij} - 1)^2 p_{ij}$.

Joint Distribution – Optimization Problem – I

Ansatz: For a risk-neutral distribution q let $\phi_{ij} := q_{ij}/\sqrt{p_{ij}}$

$$\Rightarrow \sigma^2(M) = \sum_{i,j} \left(\frac{q_{ij}}{p_{ij}} - 1 \right)^2 p_{ij} = \sum_{i,j} \phi_{ij}^2 - 1 \tag{42}$$

We then formulate the following constrained optimization problem:

$$\min_{\phi:\phi_{ij}>0} \sum_{i,j} \phi_{ij}^2 \ ,$$

s.t.

$$\sum_j \phi_{i,j}\sqrt{p_{i,j}} = \mu_i$$

$$\sum_i \phi_{i,j}\sqrt{p_{i,j}} = \nu_j$$

$$\phi_{i,j} \geq 0$$

where μ_i, ν_j represent the marginal distribution of token (i,j). details

Joint Distribution – Optimization Problem – II

Inputs:

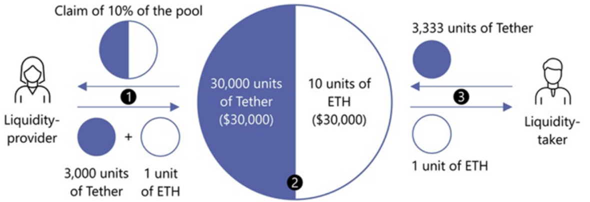
- Risk-neutral density for the margins (q_i) (obtained from constructed option surfaces).
- Real-world density for the margins (p_i) (obtained from realized data: bivariate Gaussian estimated over a rolling window of 30 days).
- Remark: We reformulate the problem in vector/matrix to solve it. For $n = m = 100$ we obtain 200 constraints involving an $n \times m$ (10000) sparse matrix. We resort to the Matlab large sparse QP solver.

AMM Arithmetics

AMM Arithmetics

DEX – constant product function market-maker — Graph A

0 *Suppose 1 Tether = $1, 1 ETH = $3,000*
Initially, 27,000 units of Tether and 9 units of ETH, each worth $27,000



Transactions:

1. *A liquidity-provider "deposits" 3,000 units of Tether and 1 unit of ETH*

2. *After this deposit, the pool contains 30,000 units of Tether and 10 units of ETH*
*Following the bonding curve, the constant equals 300,000 (= 30,000 * 10)*
The liquidity-provider has a claim of 10% of the pool's crypto-assets

3. *A liquidity-taker wishes to buy 1 unit of ETH*
*The price for 1 ETH is (30,000 + x) * (10 – 1) = 300,000*
Thus, x = 3,333, which is the amount of Tether the taker pays for 1 ETH
End result: the pool contains 33,333 units of Tether and nine units of ETH

Liquidity-provider's value
(10% of the pool)

Before trade
3,000 units of Tether
+ 1 unit of ETH
= $6,000

After trade
3,333 units of Tether
+ 0.9 units of ETH
= $6,033

Source: Authors' elaboration.

Figure: **AMM Functioning**

Data

Options – Deribit – I

- Deribit: so far the largest exchange for options on cryptocurrencies

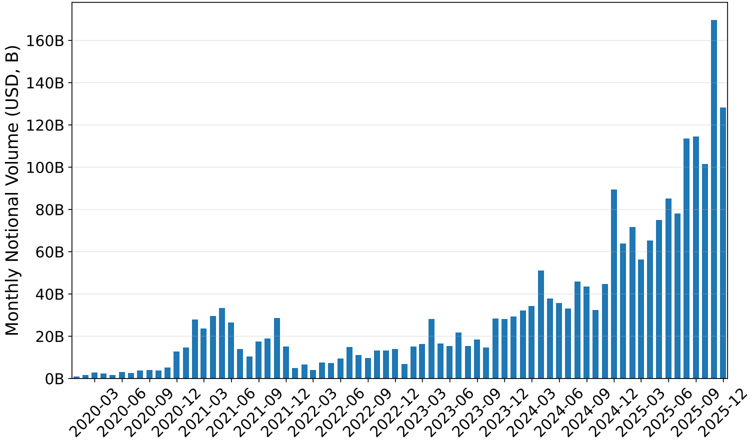


Figure: **Monthly Option Volumes on Deribit - BTC.** The figure reports the monthly total option trading volume on Deribit, measured in notional value (USD). Notional volume is defined as the total value of option contracts traded, including both on-screen trades executed directly on the exchange and block trades executed through third-party platforms. The data are sourced from Amberdata.

Options – Deribit – II

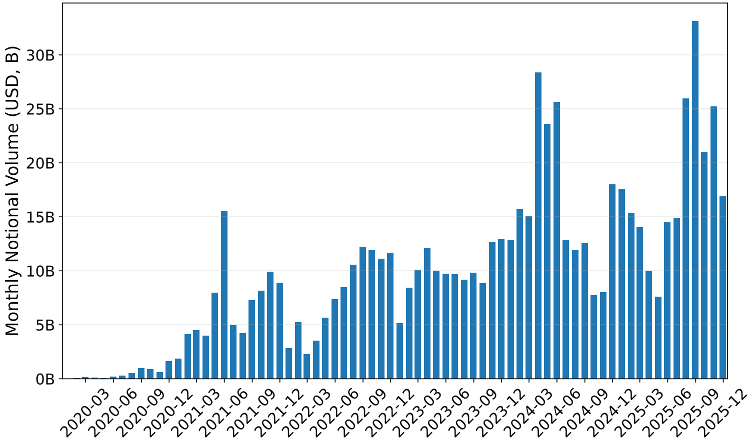


Figure: **Monthly Option Volumes on Deribit - ETH.** The figure reports the monthly total option trading volume on Deribit, measured in notional value (USD). Notional volume is defined as the total value of option contracts traded, including both on-screen trades executed directly on the exchange and block trades executed through third-party platforms. The data are sourced from Amberdata.

DeFi and Yield Farming – Ethereum